A GEOMETRIC CHARACTERIZATION OF THE DIRAC DUAL DIRAC METHOD

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Abstract. Let G be a discrete, torsion free group with a finite dimensional classifying space BG. We show that the existence of a γ-element for such G is a metric, that is, coarse, invariant of G. We also obtain results for groups with torsion. The method of proof involves showing that a group G possesses a γ-element if and only if a certain coarse (co)-assembly map is an isomorphism.

1. Introduction

The *Descent Principle* asserts that the strong Novikov conjecture for a discrete group G with finite classifying space BG may be deduced from a statement that concerns only the large scale geometry of G. In its most common formulation:

Theorem 1 (Descent)**.** *Let G be a discrete group with finite classifying space BG. If the coarse Baum-Connes assembly map $\mu_X\colon \mathrm{KX}_*(G) \to \mathrm{K}_*(C^*(|G|))$ is an isomorphism, then the assembly map $\mu\colon \mathrm{K}_*(BG) \to \mathrm{K}_*(C^*_{\mathrm{r}}(G))$ is injective.*

In this paper, we arrive at a new and somewhat strengthened version of the Principle of Descent which deals with the existence of a γ-element.

We first recall how the notion of γ-element arose in the work of Kasparov in [8]. Suppose that G is the fundamental group of a compact, aspherical, even dimensional spin$^{\mathrm{c}}$-manifold M and consider the class $\mathcal{D} = [\not{D}]$ in $\mathrm{K}_0(M) \cong \mathrm{KK}^G(C_0(\tilde{M}), \mathbb{C})$ of the Dirac operator on M. If A is any G-C^*-algebra, taking the Kasparov product of $\mathcal{D}$ with 1_A, one obtains a class $\mathcal{D} \otimes 1_A \in \mathrm{KK}(C_0(\tilde{M}) \otimes A \rtimes_{\mathrm{r}} G, A \rtimes_{\mathrm{r}} G)$. This class induces a map $\mathrm{K}_*\bigl(C_0(\tilde{M}) \otimes A \rtimes_{\mathrm{r}} G\bigr) \to \mathrm{K}_*(A \rtimes_{\mathrm{r}} G)$, which can be identified with the Baum-Connes assembly map for G with coefficients in A. Under certain geometric hypotheses, there exists $\eta \in \mathrm{KK}^G(\mathbb{C}, C_0(\tilde{M}))$ such that $\mathcal{D} \otimes_{\mathbb{C}} \eta = 1_{C_0(\tilde{M})}$. It follows that the Baum-Connes assembly map is *split injective*, and hence that the Novikov conjecture holds for any group G satisfying these geometric hypotheses. This method of verifying the Novikov conjecture has since become known as the *Dirac dual Dirac method*.

It is shown in [13] that a substitute $\mathcal{D} \in \mathrm{KK}^G(\mathcal{P}, \mathbb{C})$ for the class of the Dirac operator in the above argument exists for any locally compact group G. This class, or as we will refer to it, *morphism*, is constructed using general results on triangulated categories, and called a *Dirac morphism* for G. It is interpreted as a *projective resolution* of $\mathbb{C}$ in the category KK^G, with respect to a certain localizing subcategory. For any G-C^*-algebra A, the morphism $\mathcal{D} \otimes 1_A$ induces a map $\mathrm{K}_*(\mathcal{P} \otimes A \rtimes_{\mathrm{r}} G) \to \mathrm{K}_*(A \rtimes_{\mathrm{r}} G)$, and this map is naturally isomorphic to the Baum-Connes assembly map. This means that the left hand side $\mathrm{K}^{\mathrm{top}}(G; A)$ of the Baum-Connes assembly map may be regarded as the *left derived functor* of the right hand side, $\mathrm{K}_*(A \rtimes_{\mathrm{r}} G)$. Projective resolutions are unique up to the obvious notion of KK^G-equivalance, so that $\mathcal{D}$ is canonically associated to the group G.

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As a consequence, the Baum-Connes assembly map is an isomorphism with arbitrary coefficients if $\mathcal{D}$ is invertible in $\mathrm{KK}^G(\mathcal{P},\mathbb{C})$, and split injective with arbitrary coefficients if there exists $\eta \in \mathrm{KK}^G(\mathbb{C},\mathcal{P})$ with $\mathcal{D} \otimes_{\mathbb{C}} \eta = 1_{\mathcal{P}}$. Such an η is called a *dual Dirac morphism* and $\gamma = \eta \otimes_{\mathcal{P}} \mathcal{D}$ a *γ-element* for G. The dual Dirac morphism and the γ-element are unique if they exist. If the Dirac dual Dirac method applies to G in the sense of, say, [5], then there exists a dual Dirac morphism in the above sense. The converse also holds if the domain $\mathcal{P}$ of the Dirac morphism can be taken to be proper. This is satisfied for all groups which we consider in this article.

It is rather easy to see that a dual Dirac morphism exists if and only if the Dirac morphism induces an isomorphism

$$\mathcal{D} \otimes \cdot \colon \mathrm{KK}^G(\mathbb{C},\mathcal{P}) \to \mathrm{KK}^G(\mathcal{P},\mathcal{P})$$

If G is discrete with a finite model for its classifying space BG then this is the case if and only if the Dirac morphism induces an isomorphism

$$\mathcal{D} \otimes \cdot \colon \mathrm{KK}^G(\mathbb{C}, C_0(G)) \to \mathrm{KK}^G(\mathcal{P}, C_0(G)).$$

We call this map the *analytic co-assembly map*. The main observation of this article is that this map is a metric invariant of G.

To show this, we begin by identifying the domain $\mathrm{KK}^G(\mathbb{C}, C_0(G))$ of the above map with the reduced K-theory of a certain C^*-algebra $\mathfrak{c}(G)$ which is a variant of the Higson corona of G and that only depends on the structure of G as a coarse space and not on the algebraic structure of G as a group. We call $\mathfrak{c}(G)$ the *stable Higson corona* of G.

The right hand side of the map above can be computed via a spectral sequence with $E^2_{pq} = H^p(G,\mathbb{Z}G)$ for even q and $E^2_{pq} = 0$ for odd q and with $d^2_{pq} = 0$. It is known that the group $H^*(G,\mathbb{Z}G)$ is isomorphic to the coarse cohomology $\mathrm{HX}^*(G)$ of G as defined in [15] and in particular depends only on the structure of G as a coarse space. Suppose now that there exists a compact model for the classifying space BG. This implies that G is torsion free and that G is coarsely equivalent to $\mathcal{E}G$. Hence $\mathrm{HX}^*(G) \cong \mathrm{HX}^*(\mathcal{E}G)$. Since $\mathcal{E}G$ is uniformly contractible, $\mathrm{HX}^*(\mathcal{E}G) \cong H^*_{\mathrm{c}}(\mathcal{E}G)$, which is rationally isomorphic to $\mathrm{K}^*(\mathcal{E}G)$. These calculations suggest that $\mathrm{KK}^G(\mathcal{P}, C_0(G)) \cong \mathrm{K}^*(\mathcal{E}G)$, and this is indeed the case.

By construction of the stable Higson corona there is a connecting map

$$\mu^* \colon \tilde{\mathrm{K}}_{*+1}(\mathfrak{c}(G)) \xrightarrow{\cong} \tilde{\mathrm{K}}_{*+1}(\mathfrak{c}(\mathcal{E}G)) \to \mathrm{K}^*(\mathcal{E}G).$$

We call it the *coarse co-assembly map for G* due to its duality, discussed in [4], with the ordinary coarse Baum-Connes assembly map appearing in Theorem 1. To summarize, there are natural isomorphisms for which the following diagram commutes:

$$\begin{tikzcd}
\mathrm{KK}^G_*(\mathbb{C}, C_0(G)) \arrow[r, "\mathcal{D}\otimes\cdot"] \arrow[d, "\cong"] & \mathrm{KK}^G_*(\mathcal{P}, C_0(G)) \arrow[d, "\cong"] \\
\tilde{\mathrm{K}}_{*+1}(\mathfrak{c}(G)) \arrow[r, "\mu^*"] & \mathrm{K}^*(\mathcal{E}G).
\end{tikzcd}$$

The bottom map in this diagram is coarse, and the top determines whether or not G has a dual Dirac morphism. We have, therefore:

Theorem 2. *Let G be a discrete group with finite BG. Then G has a dual Dirac morphism if and only if the coarse co-assembly map $\mu^* \colon \tilde{\mathrm{K}}_{*+1}(\mathfrak{c}(G)) \to \mathrm{K}^*(\mathcal{E}G)$ is an isomorphism.*

When G does not possess a finite model for BG it becomes necessary to coarsen the K-theory group appearing as the target of the coarse co-assembly map μ^*. We define the *coarse* K*-theory of G* by $\mathrm{KX}^*(G) \overset{\mathrm{def}}{=} \mathrm{K}_*\bigl(\varprojlim C_0(P_dG)\bigr)$, where P_dG is the

Rips complex for G of parameter d and $\varprojlim C_0(P_d G)$ is the σ-C^*-algebra associated to the projective system $C_0(P_{d+1}G) \to C_0(P_d G)$. K-theory for σ-C^*-algebras is defined by N. C. Phillips in [14]. We are also obliged to introduce coefficients into these groups and maps, but this is easily done, and in this way one obtains a *coarse co-assembly map with coefficients in a C^*-algebra D*,

$$\mu_D^*\colon \tilde{\mathrm{K}}_{*+1}(\mathfrak{c}(G, D)) \to \mathrm{KX}^*(G, D).$$

Theorem 3. *Let G be a discrete, torsion free group with finite dimensional model for BG. Then G has a dual Dirac morphism if and only if the coarse co-assembly map μ_D^* with coefficients in D is an isomorphism for every C^*-algebra D.*

In fact, it suffices to assume that μ_D^* is an isomorphism for $D = C_0(\mathbb{N})$.

Corollary 4. *Let G be a discrete, torsion free group with a finite dimensional model for BG. Then the existence or non-existence of a γ-element for G is geometric, that is, it only depends on the large scale geometry of G. In particular, if two such groups G and G' are coarsely equivalent, then G has a γ-element if and only if G' does.*

The hypothesis of finite dimensional BG is needed because we use a concrete model for the Dirac morphism constructed by G. Kasparov and G. Skandalis in [9]. It would be more in the spirit of [13] to work with the abstract definition of the Dirac morphism. However, issues with countably infinite direct sums make such a line of argument technically difficult. The problem only occurs in the passage from the analytic co-assembly map to the existence of a dual Dirac morphism, which involves a Mayer-Vietoris argument (see Section 3).

We also investigate the case in which G has torsion. To do so, we work equivariantly with respect to finite subgroups of G. Any such subgroup $H \subseteq G$ acts on $\mathfrak{c}(G)$, so that we can form the crossed product $\mathfrak{c}(G) \rtimes H$. More generally, if D is an H-C^*-algebra, then H acts on $\mathfrak{c}(G, D)$. We construct an *H-equivariant coarse co-assembly map with coefficients in D*

$$\mu_{D,H}^*\colon \tilde{\mathrm{K}}_{*+1}(\mathfrak{c}(G, D) \rtimes H) \to \mathrm{KX}_H^*(G, D).$$

This map depends only on the H-equivariant coarse equivalence class of G. If G possesses a γ-element, then $\mu_{D,H}^*$ is an isomorphism for all D and H. With our current methods, we can prove the converse under some hypotheses on G:

Theorem 5. *Let G be a discrete group with a finite dimensional model for $\mathcal{E}G$, and assume that G has only finitely many conjugacy classes of finite subgroups. (This occurs for instance if G has a G-finite model for $\mathcal{E}G$.) Then G has a dual Dirac morphism if and only if the H-equivariant coarse co-assembly map with coefficients in D is an isomorphism for every finite subgroup H and every H-C^*-algebra D.*

Once again, it suffices to require an isomorphism for $D = C_0(\mathbb{N})$.

Let G be the fundamental group of a compact, aspherical manifold M. Then the K-theory of $\tilde{M}$ is generated by the Bott class $\beta \in \mathrm{K}^{-n}(\tilde{M})$. Theorem 5 implies that G has a dual Dirac morphism if and only if there is a unique θ in the K-theory of $\mathfrak{c}(G)$ with $\partial(\theta) = \beta$. If this is the case, then the dual Dirac morphism can be manufactured in finitely many steps from θ, each of them using a Mayer-Vietoris argument. The condition that $\partial(\theta) = \beta$ Poincaré dualizes to ensure that $\mathcal{D} \otimes \eta = 1$. Similar remarks hold for general (torsion free) discrete groups G with finite-dimensional $\mathcal{E}G$. Thus, a dual-Dirac morphism for a group G must, if it exists, arise from K-theory classes for the stable Higson corona of G.

The coarse co-assembly map for a general coarse space X is studied in more detail in [4], where we give several cases in which it is an isomorphism. This is the case for scaleable spaces in the sense of [7] and for spaces which uniformly embed

in a Hilbert space. The first result implies that a torsion free group that is coarsely equivalent to a uniformly contractible, scaleable space, has a γ-element. Such a γ-element *a priori* does not arise from any of the usual methods of constructing them (for example from Lipschitz maps to Euclidean space).

2. Projective resolutions, Dirac and dual Dirac morphisms

We shall use some ideas of [13], in which the role of the Dirac operator is centralized in the construction of the Baum-Connes assembly map as a derived functor. We begin by recalling these results.

Let G be a locally compact group, H a compact subgroup of G, and D an H-C^*-algebra. One has two functors: the *restriction functor* $\operatorname{Res}_G^H\colon \mathrm{KK}^G \to \mathrm{KK}^H$, whose definition is obvious, and the *induction functor* $\operatorname{Ind}_H^G\colon \mathrm{KK}^H \to \mathrm{KK}^G$. The latter is defined on objects, that is, H-C^*-algebras, by setting

$$\operatorname{Ind}_H^G D \overset{\text{def}}{=} \{f \in C_0(G,D) \mid \alpha_h\bigl(f(g)\bigr) = f(gh) \text{ for all } h \in H,\ g \in G\},$$

with G-action $(gf)(g') = f(g^{-1}g')$. Similarly one defines $\operatorname{Ind}_H^G(\mathcal{E})$ for H-equivariant Hilbert (bi)modules. We call an object of KK^G *compactly induced* if it is KK^G-equivalent to a G-C^*-algebra of the form $\operatorname{Ind}_H^G D$. Let $\mathcal{CI} \subseteq \mathrm{KK}^G$ denote the class of all compactly induced objects, and let $\langle\mathcal{CI}\rangle \subseteq \mathrm{KK}^G$ be the localizing subcategory generated by $\mathcal{CI}$. This is, by definition, the smallest full subcategory of KK^G satisfying:

(1) $\langle\mathcal{CI}\rangle$ contains $\mathcal{CI}$;
(2) $\langle\mathcal{CI}\rangle$ is triangulated, that is, closed under suspensions and under extensions with an equivariant, completely positive section;
(3) $\langle\mathcal{CI}\rangle$ is closed under countable direct sums.

The subcategory $\langle\mathcal{CI}\rangle$ contains all proper G-C^*-algebras (see [13]). The Baum-Connes assembly map is an isomorphism for coefficients in $\langle\mathcal{CI}\rangle$.

An element $f \in \mathrm{KK}^G(A,B)$ is called a *weak equivalence* if $\operatorname{Res}_G^H(f)$ is invertible in $\mathrm{KK}^H(A,B)$ for all compact subgroups $H \subseteq G$. An object $A \in \mathrm{KK}^G$ is called *weakly contractible* if $\operatorname{Res}_G^H(A) \cong 0$. Equivalently, the zero map $0 \to A$ is a weak equivalence. We let $\mathcal{CC} \subseteq \mathrm{KK}^G$ be the full subcategory of weakly contractible objects.

The idea of [13] is to mimic the construction of derived functors and categories in homological algebra using the above definitions. This leads to a sort of dictionary in which the weakly contractible objects play the role of the exact chain complexes, the weak equivalences play the role of the quasi-isomorphisms, and the objects of $\langle\mathcal{CI}\rangle$ play the role of the projective chain complexes. Thus the analogue of a projective resolution of $A \in \mathrm{KK}^G$ is a weak equivalence $P \to A$ with $P \in \langle\mathcal{CI}\rangle$. The case $A = \mathbb{C}$ is especially important:

Definition 6. A *Dirac morphism* for G is a weak equivalence $\mathcal{D} \in \mathrm{KK}^G(\mathcal{P},\mathbb{C})$ with $\mathcal{P} \in \langle\mathcal{CI}\rangle$.

In [13], the following facts are established:

Theorem 7. *Let G be a locally compact group.*

(1) A Dirac morphism for G exists and is unique up to KK^G-equivalence.
(2) The Baum-Connes assembly map with coefficients in A is naturally isomorphic to the map

$$\mathcal{D}_*\colon \mathrm{K}_*((\mathcal{P} \otimes A) \rtimes_{\mathrm{r}} G) \to \mathrm{K}_*(A \rtimes_{\mathrm{r}} G),$$

induced by a Dirac morphism $\mathcal{D} \in \mathrm{KK}^G(\mathcal{P},\mathbb{C})$.

(3) *Let $\mathcal{E}G$ be a locally compact model for the universal proper G-space and let $\mathcal{D} \in \mathrm{KK}^G(\mathcal{P}, \mathbb{C})$ be a Dirac morphism. For every pair of G-C^*-algebras A and B there is a natural isomorphism*
$$\delta_{AB}\colon \mathrm{KK}^G(\mathcal{P} \otimes A, B) \cong \mathrm{RKK}^G(\mathcal{E}G; A, B)$$
such that the following diagram commutes:
$$\begin{tikzcd} \mathrm{KK}^G(A,B) \arrow[r, "\delta_{AB}", "\cong"'] & \mathrm{RKK}^G(\mathcal{E}G; A, B) \\ \mathrm{KK}^G(A,B), \arrow[u, "\mathcal{D}\otimes\cdot"] \arrow[ur, "p^*_{\mathcal{E}G}"'] & \end{tikzcd}$$
where $p^*_{\mathcal{E}G}\colon \mathrm{KK}^G(A,B) \to \mathrm{RKK}^G(\mathcal{E}G; A, B)$ is induced by the map $\mathcal{E}G \to \star$, where $\star$ is a point.

Statement (1) asserts that projective resolutions exist for all $A \in \mathrm{KK}^G$. Statement (2) implies that the functor $A \mapsto \mathrm{K}^{\mathrm{top}}(G, A)$ is the left derived functor of the functor $A \to \mathrm{K}(A \rtimes_{\mathrm{r}} G)$ and statement (3) means that the category $\mathrm{RKK}^G(\mathcal{E}G)$ is the derived category of KK^G with respect to the weak equivalences. The third statement also allows us to give the following characterization of the Dirac morphism, which will be of use to us.

Lemma 8. *Let G be a locally compact group, let A be a G-C^*-algebra and $d \in \mathrm{KK}^G(A, \mathbb{C})$. Then d is a Dirac morphism for G if and only if for every G-C^*-algebra B there is a natural isomorphism*
$$\delta_B\colon \mathrm{KK}^G(A, B) \cong \mathrm{RKK}^G(\mathcal{E}G; \mathbb{C}, B)$$
such that the following diagram commutes:

(1)
$$\begin{tikzcd} \mathrm{KK}^G(A,B) \arrow[r, "\delta_{B}", "\cong"'] & \mathrm{RKK}^G(\mathcal{E}G; \mathbb{C}, B) \\ \mathrm{KK}^G(\mathbb{C},B). \arrow[u, "d\otimes\cdot"] \arrow[ur, "p^*_{\mathcal{E}G}"'] & \end{tikzcd}$$

Proof. It is shown in [13] that a Dirac morphism $\mathcal{D} \in \mathrm{KK}^G(\mathcal{P}, \mathbb{C})$ has these properties (compare (3) in Theorem 7). Conversely, the hypotheses on d determine the functor $B \mapsto \mathrm{KK}^G(A, B)$ and the natural transformation $d \otimes \cdot\colon \mathrm{KK}^G(\mathbb{C}, B) \to \mathrm{KK}^G(A, B)$ uniquely. By the Yoneda Lemma, this implies that $A \cong \mathcal{P}$ and that d corresponds to $\mathcal{D}$ under this isomorphism. □

The isomorphism (1) holds for the case of the classical Dirac operator on a manifold (see [8, Theorem 4.9]) and for the Kasparov-Skandalis Dirac morphism associated to a finite dimensional simplicial complex (see [9, Theorem 6.5]). Thus we have the following.

Corollary 9. *Let G be a locally compact group.*

A. *If $\mathcal{E}G$ can be realized by a complete Riemmanian manifold on which G acts properly and isometrically, then the class $[\mathcal{D}_X] \in \mathrm{KK}^G(C_\tau(X), \mathbb{C})$ of [8, Definition 4.2] is a Dirac morphism for G.*

B. *If $\mathcal{E}G$ can be realized by a finite dimensional simplicial complex X on which G acts simplicially, then the Dirac morphism for G may be identified with the class $[\mathcal{D}_X] \in \mathrm{KK}^G(\mathcal{P}_X, \mathbb{C})$ of [9, Definition 1.3].*

Note 10. In Corollary 9.B we have changed notation from [9], denoting by $\mathcal{P}_X$ what they have denoted by $\mathcal{A}_X$.

Remark 11. Formally, the source $\mathcal{P}$ of the Dirac morphism must be an ungraded G-C^*-algebra because the KK^G-category of graded C^*-algebras is not triangulated. However, it is certainly permissible to use a graded G-C^*-algebra that is KK^G-equivalent to an ungraded one. It is well-known that the G-C^*-algebra $C_\tau(X)$ in Corollary 9.A is KK^G-equivalent to $C_0(T^*M)$. A similar ungraded model for $\mathcal{P}_X$ is constructed in [9]. Therefore, we may ignore this technical issue in the following.

Definition 12. Let G be a locally compact group and let $\mathcal{D} \in \mathrm{KK}^G(\mathcal{P}, \mathbb{C})$. A *dual Dirac morphism* for G is an element $\eta \in \mathrm{KK}^G(\mathbb{C}, \mathcal{P})$ such that $\mathcal{D} \otimes_{\mathbb{C}} \eta = 1_{\mathcal{P}}$. The composition $\gamma = \eta \otimes_{\mathcal{P}} \mathcal{D}$ is called a *γ-element* for G.

By definition, G has a γ-element if and only if G has a dual Dirac morphism. Moreover, if a dual Dirac morphism exists, then it is unique. Consequently the same is true of γ-elements.

Remark 13. It is shown in [13] that a dual Dirac morphism exists if and only if KK^G is isomorphic as a triangulated category to the product of the subcategory $\langle \mathcal{CI} \rangle$ and the subcategory of weakly contractible objects $\mathcal{CC}$. For $A \in \mathrm{KK}^G$, let $\gamma_A \overset{\text{def}}{=} \gamma \otimes_{\mathbb{C}} 1_A \in \mathrm{KK}^G(A, A)$. Then $A \in \langle \mathcal{CI} \rangle$ if and only if $\gamma_A = 1_A$, and $A \in \mathcal{CC}$ if and only if $\gamma_A = 0$. Thus the existence of a dual Dirac morphism implies that there is no interaction between $\langle \mathcal{CI} \rangle$ and $\mathcal{CC}$, a situation which we note has no analogue in homological algebra.

The following is also shown in [13] and follows easily from Remark 13.

Proposition 14. *If a dual Dirac morphism exists, then the map*

$$p_{\mathcal{E}G}^* \colon \mathrm{KK}^G(A, B) \to \mathrm{RKK}^G(\mathcal{E}G; A, B)$$

is an isomorphism for all $A \in \mathrm{KK}^G$, $B \in \langle \mathcal{CI} \rangle$.

For convenience, we introduce the following definition.

Definition 15. The *analytic co-assembly map with coefficients in A* is the map

$$p_{\mathcal{E}G}^* \colon \mathrm{KK}^G(\mathbb{C}, A) \to \mathrm{RKK}^G(\mathcal{E}G; \mathbb{C}, A), \tag{2}$$

where $p_{\mathcal{E}G} \colon \mathcal{E}G \to \star$ is the constant map from $\mathcal{E}G$ to a point.

Proposition 16. *Let G be a locally compact group and $\mathcal{D} \in \mathrm{KK}^G(\mathcal{P}, \mathbb{C})$ a Dirac morphism for G. Then G possesses a dual Dirac morphism if and only if the analytic co-assembly map with coefficients in $\mathcal{P}$*

$$p_{\mathcal{E}G}^* \colon \mathrm{KK}^G(\mathbb{C}, \mathcal{P}) \to \mathrm{RKK}^G(\mathcal{E}G; \mathbb{C}, \mathcal{P})$$

is an isomorphism.

Proof. By statement (3) of Theorem 7, this map is an isomorphism if and only if the map

$$\mathcal{D}^* \colon \mathrm{KK}^G(\mathbb{C}, \mathcal{P}) \to \mathrm{KK}^G(\mathcal{P}, \mathcal{P})$$

is an isomorphism. If this is the case, then the inverse image η of $1_{\mathcal{P}} \in \mathrm{KK}^G(\mathcal{P}, \mathcal{P})$ is a dual Dirac morphism. The converse implication is contained in Proposition 14, since by definition $\mathcal{P} \in \langle \mathcal{CI} \rangle$.

□

Remark 17. Another definition of a γ-element that is used frequently is the following (see [5]). An element $\gamma \in \mathrm{KK}^G(\mathbb{C}, \mathbb{C})$ is called a γ-element if there is a factorization $\gamma = \alpha \otimes_P \beta$, where P is a proper G-C^*-algebra, $\alpha \in \mathrm{KK}^G(\mathbb{C}, P)$, $\beta \in \mathrm{KK}^G(P, \mathbb{C})$, and $\beta \otimes_{\mathbb{C}} \alpha = 1_P$. If such P, α and β exist, then α is necessarily a weak equivalence and hence a model for the Dirac morphism of G. Hence β is a dual Dirac morphism in our sense, and likewise for γ. We do not, however, know whether the converse is

true. That is, we do not know whether the source $\mathcal{P}$ of the Dirac morphism can be realized in general by a proper G-C^*-algebra. Hence our definition is *a priori* less restrictive than that of [5].

3. Geometrically finite groups

Propositions 14 and 16 show that a dual Dirac morphism exists if and only if the analytic co-assembly map (2) is an isomorphism for all coefficients in $\langle\mathcal{CI}\rangle$ or, equivalently, for the fixed coefficient $\mathcal{P}$, where $\mathcal{D} \in \mathrm{KK}^G(\mathcal{P}, \mathbb{C})$ is a Dirac morphism. Since the category $\langle\mathcal{CI}\rangle$ is by definition generated by $\mathcal{CI}$, it seems plausible that it suffices to check the isomorphism on objects of $\mathcal{CI}$. However, since we know nothing about the behavior of KK^G under infinite direct sums in the second variable, there is a difficulty in passing from $\mathcal{CI}$ to $\langle\mathcal{CI}\rangle$. To avoid this difficulty, we restrict attention to groups that satisfy some finiteness hypotheses that allow us to construct the domain $\mathcal{P}$ of the Dirac morphism from compactly induced G-C^*-algebras without using infinite direct sums. This technique already covers several cases of interest.

Definition 18. Let G be a discrete, countable group.

A. We say that G is *geometrically finite* if $\mathcal{E}G$ can be realized as a finite dimensional simplicial complex.
B. We say that G is *strongly geometrically finite* if it is geometrically finite and, in addition, has at most finitely many conjugacy classes of finite subgroups.

Remark 19. Clearly, if G has a G-finite model for $\mathcal{E}G$, then G is strongly geometrically finite, but the condition of strong geometric finiteness is obviously much weaker.

Theorem 20. *Let G be a discrete group.*

A. *If G is strongly geometrically finite, then G has a γ-element if and only if the analytic co-assembly map* (2) *with coefficients in $\mathrm{Ind}_H^G D$ is an isomorphism for every finite subgroup $H \subseteq G$ and every H-C^*-algebra D.*
B. *If G has a G-finite model for $\mathcal{E}G$, then G has a γ-element if and only if the analytic co-assembly map* (2) *with coefficients in $C_0(G/H)$ is an isomorphism for every finite subgroup $H \subseteq G$.*

Remark 21. The proof shows that in case A a γ-element already exists if the analytic co-assembly map is an isomorphism for $C_0(G/H \times \mathbb{N}) \cong \mathrm{Ind}_H^G C_0(\mathbb{N})$ for any H. Thus we only need one very simple coefficient algebra. Since $\mathbb{C}$ is a direct summand of $C_0(\mathbb{N})$, the analytic co-assembly map for $C_0(G/H)$ is a direct summand of the analytic co-assembly map for $C_0(G/H \times \mathbb{N})$. Therefore, if we have an isomorphism for $C_0(G/H \times \mathbb{N})$, then we also have an isomorphism for $\mathbb{C}$.

Proof. We have already observed in Proposition 16 that the existence of a γ-element implies that the analytic co-assembly map is an isomorphism for all coefficients in $\langle\mathcal{CI}\rangle$, without any hypothesis on the group G. We have to prove the converse. Let G be geometrically finite, let X be a finite dimensional simplicial complex realizing $\mathcal{E}G$, and let $[\mathcal{D}_X] \in \mathrm{KK}^G(\mathcal{P}_X, \mathbb{C})$ be the Kasparov-Skandalis realization of the Dirac morphism for G. By Proposition 16 and Corollary 9, G possesses a γ-element if and only if the analytic co-assembly map with coefficients in $\mathcal{P}_X$ is an isomorphism. Recall from [9] that the skeletal filtration of X gives rise to a filtration of $\mathcal{P}_X$ by ideals

$$0 = \mathcal{P}_X^{(-1)} \subset \mathcal{P}_X^{(0)} \subset \mathcal{P}_X^{(1)} \subset \mathcal{P}_X^{(2)} \subset \cdots \mathcal{P}_X^{(n)} = \mathcal{P}_X.$$

Since the resulting extensions

$$0 \to \mathcal{P}_X^{(k-1)} \to \mathcal{P}_X^{(k)} \to \mathcal{P}_X^{(k)}/\mathcal{P}_X^{(k-1)} \to 0$$

have G-equivariant completely positive sections, $\mathcal{P}_X$ lies in the triangulated subcategory of KK^G that is generated by the subquotients $\mathcal{P}_X^{(k)}/\mathcal{P}_X^{(k-1)}$. The latter are KK^G-equivalent to $C_0(X^{(k)})$, where $X^{(k)}$ denotes the set of k-cells of X, viewed as a discrete G-space. It follows that the analytic co-assembly map is an isomorphism with coefficients in $\mathcal{P}_X$ if it is an isomorphism with coefficients in $C_0(X^{(k)})$ for all $k \in \mathbb{N}$. Another way of expressing this is as follows. The extensions above give rise to commutative diagrams with exact rows:

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{KK}^G_*(\mathbb{C}, \mathcal{P}_X^{(k-1)}) \arrow[r] \arrow[d] & \mathrm{KK}^G_*(\mathbb{C}, \mathcal{P}_X^{(k)}) \arrow[r] \arrow[d] & \mathrm{KK}^G_*(\mathbb{C}, \mathcal{P}_X^{(k)}/\mathcal{P}_X^{(k-1)}) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \mathcal{P}_X^{(k-1)}) \arrow[r] & \mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \mathcal{P}_X^{(k)}) \arrow[r] & \mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \mathcal{P}_X^{(k)}/\mathcal{P}_X^{(k-1)}) \arrow[r] & {},
\end{tikzcd}$$

where the vertical maps are induced by product with the Dirac morphism $[\mathcal{D}_X]$. If we apply the Five Lemma to these diagrams, we obtain by induction on k that the analytic co-assembly map is an isomorphism with coefficients $\mathcal{P}_X^{(k)}$ for $k = -1, \dots, n$ provided it is an isomorphism for the subquotients. By the identification of the subquotients above, this will be the case if the analytic co-assembly map with coefficients $C_0(X^{(k)})$ is an isomorphism for all k.

Each of the discrete, proper G-spaces $X^{(k)}$ is G-isomorphic to a countably infinite disjoint union of homogeneous spaces G/H for finite subgroups $H \subseteq G$. Since G has only finitely many conjugacy classes of finite subgroups, at most finitely many of these homogeneous spaces are non-isomorphic as G-spaces. Thus $C_0(X^{(k)})$ is G-isomorphic to a *finite* direct sum of G-C^*-algebras of the form $C_0(I \times G/H)$ with finite subgroups $H \subseteq G$ and a discrete set I with trivial action of G. In this way we are reduced to verifying isomorphism of the analytic co-assembly map with coefficients $C_0(G/H \times I)$ for an at most countable set I. In case B, only finite sets I are involved, so that in this case we only need the analytic co-assembly map to be an isomorphism with coefficients $C_0(G/H)$. If the set I is infinite, we can identify it with the fixed, countable set $\mathbb{N}$ with trivial G-action, so that we only require the analytic co-assembly map to be an isomorphism with coefficients $C_0(G/H \times \mathbb{N})$. $\square$

4. The stable Higson corona

For basic matters on coarse structures we refer the reader to [7]. Such a structure on a locally compact topological space X is given by a collection of *entourages* $E \subset X \times X$ satisfying various axioms. A subset $B \subset X$ is called *bounded* (with respect to the coarse structure) if $B \times B$ is an entourage. The coarse structure is called *proper* if every bounded subset is compact, and *separable* if X has a countable, uniformly bounded open cover. Finally, we say that the coarse structure is *unital* if the diagonal is an entourage.

An example of a coarse structure satisfying all these conditions arises when X is a separable metric space of uniformly bounded geometry, and the entourages are given by sets of the form

$$E = \{(x, y) \in X \times X \mid d(x, y) \leq R\}.$$

We call a coarse structure metrizable if it arises from a metric in this fashion. Any proper, unital, separable coarse structure on a second countable space X is metrizable. In this paper we are exclusively interested in this case.

Convention: By "coarse space" we always mean a second countable, locally compact topological space equipped with a separable, proper, and unital coarse structure.

A (not necessarily continuous) map between coarse spaces is a *coarse map* if it maps entourages in X to entourages in Y and is *proper* in the sense that inverse

images of bounded subsets are bounded. Two maps ϕ and ψ from X to Y are called *close* if $\{(\phi(x), \psi(x)) \mid x \in X\}$ is an entourage in $Y \times Y$; that is, if $\phi \times \psi$ maps the diagonal in $X \times X$ to an entourage in $Y \times Y$. Finally, we say that two coarse spaces X and Y are *coarsely equivalent* if there exist coarse maps $X \to Y$ and $Y \to X$ whose compositions are close to the identity maps on X and Y, respectively. Finally, if X is a coarse space and G is a group acting by homeomorphisms of X, then a coarse structure is G-invariant if any entourage is contained in a G-invariant entourage. We also say that G acts *isometrically* on the coarse space X.

Every discrete and countable group G may be regarded as a coarse space. Its coarse structure is defined by the entourages

$$E = \{(g_1, g_2) \mid g_1 g_2^{-1} \in F\},$$

where F ranges over the finite subsets of G. This coarse structure is manifestly *right* translation invariant and is the unique coarse structure on G with this property. We always equip G with this coarse structure. The metrizability of this coarse structure means that there is a proper length function l on G such that the associated right invariant metric $d(g,h) = l(gh^{-1})$ induces the coarse structure. It is clear that any two such l induce the same coarse structure; thus the length function l is not unique, but the coarse structure it generates is.

Let $f\colon X \to E$ be a function from a coarse space X into a Banach space E. Define a function $\nabla f\colon X \times X \to \mathbb{R}_+$ by $\nabla f(x,y) = \|f(x) - f(y)\|$.

Definition 22. Let X be a coarse space and let f be a continuous function on X with values in a Banach space E. Then we say that f has *vanishing variation* if the restriction of ∇f to the closure of any entourage vanishes at infinity.

Let $\mathbb{K}$ be the C^*-algebra of compact operators on $\ell^2(\mathbb{N})$.

Definition 23. Let X be a coarse space and let D be a C^*-algebra. Then $\bar{\mathfrak{c}}(X, D)$ shall denote the C^*-algebra of bounded, continuous functions $X \to D \otimes \mathbb{K}$ of vanishing variation.

The central definition of this paper is the following.

Definition 24. Let X be a coarse space. Then the *stable Higson corona* $\mathfrak{c}(X, D)$ *of* X *with coefficients in* D is the C^*-algebra

$$\mathfrak{c}(X, D) = \bar{\mathfrak{c}}(G, D)/C_0(X, D \otimes \mathbb{K}).$$

Remark 25. It is not difficult to show that the isomorphism class of the algebra $\mathfrak{c}(X, D)$ only depends on the coarse equivalence class of X. In particular, if one replaces X by a discrete subset X' such that the inclusion $X' \to X$ is a coarse equivalence, then the restriction map $\mathfrak{c}(X, D) \to \mathfrak{c}(X', D)$ is a $*$-isomorphism. More generally, a coarse map $X \to Y$ induces a canonical $*$-homomorphism $\mathfrak{c}(Y, D) \to \mathfrak{c}(X, D)$, and two close maps induce the same homomorphism. Hence $X \mapsto \mathfrak{c}(X, D)$ is functorial from the category of coarse spaces and coarse maps to the category of C^*-algebras and $*$-homomorphisms. See [4] for details, or [15] for the analogous assertions for the Higson corona.

It is convenient for this article to replace $\mathfrak{c}(X, D)$ and $\bar{\mathfrak{c}}(X, D)$ by the following variants. We denote the multiplier algebra of a C^*-algebra D by $\mathcal{M}(D)$ and the stable multiplier algebra by $\mathcal{M}^s(D) \stackrel{\text{def}}{=} \mathcal{M}(D \otimes \mathbb{K})$.

Definition 26. Let X be a coarse space. Then $\bar{\mathfrak{c}}^{\mathfrak{red}}(X, D)$ shall denote the C^*-algebra of bounded, continuous functions of vanishing variation $f\colon X \to \mathcal{M}^s(D)$ such that $f(x) - f(y) \in D \otimes \mathbb{K}$ for all $x, y \in X$. The *reduced stable Higson corona of* X *with coefficients in* D, $\mathfrak{c}^{\mathfrak{red}}(X, D)$, is the quotient

$$\mathfrak{c}^{\mathfrak{red}}(X, D) = \bar{\mathfrak{c}}^{\mathfrak{red}}(X, D)/C_0(X, D \otimes \mathbb{K}).$$

The reason for the terminology is the following. Let i be the inclusion $D \otimes \mathbb{K} \to \bar{\mathfrak{c}}(X, D)$ that sends elements of $D \otimes \mathbb{K}$ to constant functions on X. We obtain an induced map $i_* \colon \mathrm{K}(D) \cong \mathrm{K}(D \otimes \mathbb{K}) \to \mathrm{K}(\bar{\mathfrak{c}}(X, D))$. Let $\pi \colon \bar{\mathfrak{c}}(X, D) \to \mathfrak{c}(X, D)$ be the quotient map.

Definition 27. We define the reduced K-theories of $\mathfrak{c}(X, D)$ and $\bar{\mathfrak{c}}(X, D)$ by

$$\tilde{\mathrm{K}}_*(\mathfrak{c}(X, D)) = \mathrm{K}_*(\mathfrak{c}(X, D))/(\pi \circ i)_* \mathrm{K}_*(D),$$
$$\tilde{\mathrm{K}}_*(\bar{\mathfrak{c}}(X, D)) = \mathrm{K}_*(\bar{\mathfrak{c}}(X, D))/i_* \mathrm{K}_*(D).$$

Lemma 28 (see [4])**.** *There are natural isomorphisms*

$$\tilde{\mathrm{K}}_*(\mathfrak{c}(X, D)) \cong \mathrm{K}_*(\mathfrak{c}^{\mathfrak{red}}(X, D)), \qquad \tilde{\mathrm{K}}_*(\bar{\mathfrak{c}}(X, D)) \cong \mathrm{K}_*(\bar{\mathfrak{c}}^{\mathfrak{red}}(X, D)).$$

From now on, we work exclusively with the reduced algebras $\bar{\mathfrak{c}}^{\mathfrak{red}}(X, D)$ and $\mathfrak{c}^{\mathfrak{red}}(X, D)$. However, it is the non-reduced algebras $\mathfrak{c}(X, D)$ and $\bar{\mathfrak{c}}(X, D)$ which are most obviously connected to classical constructions in coarse geometry, in particular to the Higson corona construction (see [7, 15] for its definition.) As in Remark 25, one can easily check the following lemma. Details can be found in [4].

Lemma 29. *The assignment $X \mapsto \mathfrak{c}^{\mathfrak{red}}(X, D)$ is functorial from the category of coarse spaces and coarse maps to the category of C^*-algebras and C^*-algebra homomorphisms. That is, a coarse map $X \to Y$ induces a canonical $*$-homomorphism $\mathfrak{c}^{\mathfrak{red}}(Y, D) \to \mathfrak{c}^{\mathfrak{red}}(X, D)$, and close maps induce the same $*$-homomorphism.*

Let H be a finite group acting continuously on the coarse space X and preserving the coarse structure. For compatibility with our later arguments, we let H act on the right. Suppose that H also acts on the coefficient algebra D on the left via a homomorphism $H \to \mathrm{Aut}(D)$, $h \mapsto \alpha_h$. Then we obtain an action of H on the algebra $\bar{\mathfrak{c}}^{\mathfrak{red}}(X, D)$ by $(h \cdot f)(x) = \alpha_h f(xh)$. This restricts to an action of H on the ideal $C_0(X, D \otimes \mathbb{K})$ and so descends to an action of H on $\mathfrak{c}^{\mathfrak{red}}(X, D)$. If H also acts on a coarse space Y, and $\phi \colon X \to Y$ is a coarse, proper, H-equivariant map, then the induced $*$-homomorphism $\mathfrak{c}^{\mathfrak{red}}(Y, D) \to \mathfrak{c}^{\mathfrak{red}}(X, D)$ is H-equivariant, so that we obtain a $*$-homomorphism

$$\mathfrak{c}^{\mathfrak{red}}(X, D) \rtimes H \to \mathfrak{c}^{\mathfrak{red}}(Y, D) \rtimes H.$$

Since H is finite the type of cross product we use is immaterial.

By construction, there is an exact sequence

$$0 \to C_0(X, D \otimes \mathbb{K}) \to \bar{\mathfrak{c}}^{\mathfrak{red}}(X, D) \to \mathfrak{c}^{\mathfrak{red}}(X, D) \to 0.$$

If H is a finite group acting on X and preserving the coarse structure, then we obtain an exact sequence

$$0 \to C_0(X, D \otimes \mathbb{K}) \rtimes H \to \bar{\mathfrak{c}}^{\mathfrak{red}}(X, D) \rtimes H \to \mathfrak{c}^{\mathfrak{red}}(X, D) \rtimes H \to 0.$$

This exact sequence is the origin of the *coarse co-assembly map* in Section 6.

5. Calculation of $\mathrm{KK}^G(\mathbb{C}, \mathrm{Ind}_H^G D)$

Fix a discrete group G, let H be a finite subgroup, and let D be an H-C^*-algebra. We are going to identify the group $\mathrm{KK}^G(\mathbb{C}, \mathrm{Ind}_H^G D)$ that appears in connection with γ-elements (Theorem 20) with the K-theory of the C^*-algebra $\mathfrak{c}^{\mathfrak{red}}(G, D) \rtimes H$ described in the previous section. We let H act on G by right translations. This action preserves the coarse structure and hence induces an action on the C^*-algebra $\mathfrak{c}^{\mathfrak{red}}(G, D)$. We also let $D_H \overset{\text{def}}{=} D \otimes \mathbb{K}(\ell^2 H)$, equipped with the usual action of H, and denote the H-fixed point subalgebra of $\mathfrak{c}^{\mathfrak{red}}(G, D_H)$ by $\mathfrak{c}^{\mathfrak{red}}(G, D_H)^H$.

Theorem 30. *Let G, H and D be as above. Then there exist natural isomorphisms*

$$\mathrm{KK}^G_*(\mathbb{C}, \mathrm{Ind}_H^G D) \cong \mathrm{K}_{*+1}(\mathfrak{c}^{\mathfrak{red}}(G, D_H)^H) \cong \mathrm{K}_{*+1}(\mathfrak{c}^{\mathfrak{red}}(G, D) \rtimes H).$$

Proof. Since $\mathbb{K}(\ell^2 H)$ is finite dimensional, we have $\mathfrak{c}^{\mathfrak{red}}(G, D_H) \cong \mathfrak{c}^{\mathfrak{red}}(G, D) \otimes \mathbb{K}(\ell^2 H)$. For any finite group H and any H-C^*-algebra B there is a canonical isomorphism $\big(B \otimes \mathbb{K}(\ell^2 H)\big)^H \cong B \rtimes H$. Hence we have an isomorphism

$$\mathfrak{c}^{\mathfrak{red}}(G, D_H)^H \cong \big(\mathfrak{c}^{\mathfrak{red}}(G, D) \otimes \mathbb{K}(\ell^2 H)\big)^H \cong \mathfrak{c}^{\mathfrak{red}}(G, D) \rtimes H,$$

which yields the second isomorphism of the theorem. Actually, in the proof we will exclusively work with fixed point algebras.

We now describe the cycles for $\mathrm{KK}^G_*(\mathbb{C}, \mathrm{Ind}_H^G D)$ more concretely, first for $* = 1$. These are given by pairs $(\mathcal{E}, F)$ where $\mathcal{E}$ is a G-equivariant Hilbert module over $\mathrm{Ind}_H^G D$ and $F \in \mathbb{B}(\mathcal{E})$ satisfies

$$F = F^*, \quad 1 - F^2 \in \mathbb{K}(\mathcal{E}), \quad gF - F \in \mathbb{K}(\mathcal{E}) \tag{3}$$

for all $g \in G$. Since $\mathrm{Ind}_H^G D$ is a proper G-C^*-algebra, the Equivariant Stabilization Theorem of [11] applies. Hence we can restrict attention to the case where $\mathcal{E}$ is the standard Hilbert module $\mathrm{Ind}_H^G D \otimes \ell^2(G \times \mathbb{N})$ over $\mathrm{Ind}_H^G D$. There is a natural isomorphism $\mathrm{Ind}_H^G D \otimes \ell^2(G \times \mathbb{N}) \cong \mathrm{Ind}_H^G \big(D \otimes \ell^2(H \times \mathbb{N})\big)$. Thus we may equivalently parametrize cycles for $\mathrm{KK}^G(\mathbb{C}, \mathrm{Ind}_H^G D)$ by operators on $\mathrm{Ind}_H^G(D \otimes \ell^2(H \times \mathbb{N}))$ that satisfy (3).

Elements of the Hilbert module $\mathrm{Ind}_H^G(D \otimes \ell^2(H \times \mathbb{N}))$ are functions in $C_0(G, D \otimes \ell^2(H \times \mathbb{N}))$ that satisfy $f(gh) = \alpha_h\big(f(g)\big)$ for all $g \in G$, $h \in H$. The right $\mathrm{Ind}_H^G D$-Hilbert module structure is given by pointwise multiplication and pointwise inner products. The group G acts by left translation. The operator F becomes a family of self-adjoint operators

$$F = (F_g)_{g \in G}, \qquad F_g \in \mathbb{B}(D \otimes \ell^2(H \times \mathbb{N})) \cong \mathcal{M}\big(D_H \otimes \mathbb{K}\big).$$

Since F preserves the covariance condition $f(gh) = \alpha_h\big(f(g)\big)$, we have

$$F_{gh} = \alpha_h(F_g) \tag{4}$$

for all $g \in G$, $h \in H$. Secondly, $1 - F_g^2 \in \mathbb{K}(D \otimes \ell^2(H \times \mathbb{N})) = D_H \otimes \mathbb{K}$, and the function $g \mapsto \|1 - F_g^2\|$ belongs to $C_0(G)$, whence,

$$1 - F^2 \in C_0(G, D_H \otimes \mathbb{K}) \tag{5}$$

Finally, $F_g - F_{xg} \in \mathbb{K}(D \otimes \ell^2(H \times \mathbb{N})) = D_H \otimes \mathbb{K}$ for all $g, x \in G$, and the function $g \mapsto \|F_g - F_{xg}\|$ belongs to $C_0(G)$ for all $x \in G$. Thus

$$F_g - F_{g'} \in \mathbb{K}(D \otimes \ell^2(H \times \mathbb{N})) = D_H \otimes \mathbb{K} \tag{6}$$

for all $g, g' \in G$, and

$$\lim_{g \to \infty} \|F_g - F_{xg}\| = 0. \tag{7}$$

for all $x \in G$.

We claim that (7) holds if and only if the function $g \mapsto F_g$ has vanishing variation in the sense of Definition 22. It is clear that vanishing variation implies (7). Conversely, if we do not have vanishing variation, then there exist sequences $g_n, g'_n \to \infty$ in G and $\epsilon > 0$ such that the sequence $\{(g_n, g'_n)\}$ is contained in some fixed entourage, but $\|F_{g_n} - F_{g'_n}\| \geq \epsilon$ for all n. To say that $\{(g_n, g'_n)\}$ lies in an entourage means that $g_n g_n'^{-1} \in \Sigma$ for all n, for some finite set $\Sigma \subset G$. After extracting a subsequence, we may replace Σ by a singleton $\{x\}$, so that $g'_n = x g_n$ for all n. Hence (7) is violated. This proves that (7) is equivalent to vanishing variation of F.

Equation (6) means that F belongs to $\bar{\mathfrak{c}}^{\mathfrak{red}}(G, D_H)$. Equation (4) means that $F \in \bar{\mathfrak{c}}^{\mathfrak{red}}(G, D_H)^H$. In addition, we have $F = F^*$ and (5). Let $[F]$ be the image of F in $\mathfrak{c}^{\mathfrak{red}}(G, D_H)$. Equation (5) means that $P_F \overset{\text{def}}{=} ([F] - 1)/2$ is a projection

in $\mathfrak{c}^{\mathfrak{red}}(G, D_H)^H$. Conversely, any projection in $\mathfrak{c}^{\mathfrak{red}}(G, D_H)^H$ is of the form P_F for some cycle F for $\mathrm{KK}^G(\mathbb{C}, \mathrm{Ind}_H^G D)$.

Since $\mathfrak{c}^{\mathfrak{red}}(G, D_H)^H$ is matrix stable, we do not have to adjoin matrices to compute its K-theory. The cycle F is degenerate if and only if $F_g = F_{xg}$ for all $g, x \in G$, that is, if and only if P_F is a *constant* function on G. The subalgebra of constant functions in $\bar{\mathfrak{c}}^{\mathfrak{red}}(G, D_H)^H$ is isomorphic to the stable multiplier algebra $\mathcal{M}\big(D_H \otimes \mathbb{K}\big)^H \cong \mathcal{M}(D \rtimes H \otimes \mathbb{K})$ and hence has vanishing K-theory. Two cycles F and F' satisfy $P_F = P_{F'}$ if and only if F' is a compact perturbation of F. An operator homotopy between two cycles is the same as a homotopy between the associated projections. As a result, the equivalence relation generated by addition of degenerate cycles and operator homotopy for F is equivalent to the equivalence relation of stable homotopy equivalence of projections in $\mathfrak{c}^{\mathfrak{red}}(G, D_H)^H$. Since operator homotopy and homotopy generate the same equivalence relation on $\mathrm{KK}^G(\mathbb{C}, \mathrm{Ind}_H^G D)$, we obtain $\mathrm{KK}_1^G(\mathbb{C}, \mathrm{Ind}_H^G D) \cong \mathrm{K}_0(\mathfrak{c}^{\mathfrak{red}}(G, D_H)^H)$ as claimed.

Consider now cycles for $\mathrm{KK}_0^G(\mathbb{C}, \mathrm{Ind}_H^G D)$. Thus we also have a grading on our Hilbert module $\mathcal{E}$, and F is odd. Since F is self-adjoint and odd, knowing F is equivalent to knowing its restriction $U\colon \mathcal{E}_+ \to \mathcal{E}_-$. We may assume that the even and odd parts $\mathcal{E}_\pm$ are isomorphic to $\mathrm{Ind}_H^G(D \otimes \ell^2(H \times \mathbb{N}))$. As above, we obtain $U \in \bar{\mathfrak{c}}^{\mathfrak{red}}(G, D_H)^H$ with unitary image in $\mathfrak{c}^{\mathfrak{red}}(G, D_H)^H$. Conversely, any unitary in $\mathfrak{c}^{\mathfrak{red}}(G, D_H)^H$ arises in this fashion. We have similar criteria for degenerate Kasparov cycles, compact perturbations, and operator homotopy. This yields $\mathrm{KK}_0^G(\mathbb{C}, \mathrm{Ind}_H^G D) \cong \mathrm{K}_1(\mathfrak{c}^{\mathfrak{red}}(G, D_H)^H)$ as above. □

6. The coarse co-assembly map

In this section we define coarse co-assembly maps which we will eventually identify with the analytic co-assembly maps occuring in Theorem 20. We first introduce a K-theoretic analogue of the coarse K-homology of a space X. We do this equivariantly with respect to a finite group action, and with coefficients.

Let X be a coarse space. By our convention, (in particular by the bounded geometry assumption), X is coarsely equivalent to a discrete coarse space. Moreover, if H is a finite group acting isometrically on X, then X can even be arranged to be H-equivariantly coarsely equivalent to a discrete H-space. Since our constructions depend only on the H-equivariant coarse equivalence class of X, we may assume X discrete to begin with. Let D be an H-C^*-algebra and recall that D_H denotes $D \otimes \mathbb{K}(\ell^2 H)$ with its canonical H-action.

Let E be an H-invariant entourage of X, and let $P_E(X)$ be the simplicial complex whose vertices are the points of X and whose simplices are the finite subsets $F \subseteq X$ for which $F \times F \subset E$. We do not distinguish between this simplicial complex and its geometric realization, so that $P_E(X)$ is a locally compact H-space. We view elements of $P_E(X)$ as probability measures on X in the usual fashion, so that the support of an element of $P_E(X)$ is a subset of X. We equip $P_E(X)$ with the coarse structure generated by the entourages

$$\{(\mu, \nu) \in P_n(X) \times P_n(X) \mid \mathrm{supp}(\mu) \times \mathrm{supp}(\nu) \subset F\},$$

where F ranges over the entourages of X. It is clear that the finite group H preserves this coarse structure. If E is the diagonal, we get back the space X itself. If $E_1 \subset E_2$, then there is an injective, proper, H-equivariant, continuous coarse equivalence $P_{E_1}(X) \to P_{E_2}(X)$. In particular, we obtain H-equivariant coarse equivalences $X \to P_E(X)$ for any E that contains the diagonal, and these maps are compatible with the maps $P_{E_1}(X) \to P_{E_2}(X)$.

Since X is assumed separable, we can choose an increasing sequence (E_n) of H-invariant entourages of X such that any entourage of X is contained in E_n for

some n. Let $P_n(X)$ denote $P_{E_n}(X)$ and let $i_n\colon P_n(X) \to P_{n+1}(X)$ be the canonical map. These maps are injective and H-equivariant. We denote the resulting inductive system by $\mathcal{P}(X)$. Let $|\mathcal{P}(X)|$ be the inductive limit of this system, equipped with the canonical topology. We let

$$\hat{C}_0(\mathcal{P}(X), D) \rtimes H \overset{\text{def}}{=} \varprojlim C_0(P_n(X), D) \rtimes H.$$

This is a σ-C^*-algebra in the sense of [14]. Since the maps i_n are injective, the induced maps on C_0-functions are surjective. Hence the maps $\hat{C}_0(\mathcal{P}(X), D) \to C_0(P_n(X), D)$ are surjective for all $n \in \mathbb{N}$. This yields an isomorphism

$$\hat{C}_0(\mathcal{P}(X), D) \cong \{f\colon |\mathcal{P}(X)| \to \mathbb{C} \mid f_{|P_n(X)} \in C_0(P_n(X), D) \text{ for all } n \in \mathbb{N}\}.$$

Different choices of the sequence of entourages (E_n) yield isomorphic inductive systems $\mathcal{P}(X)$ and hence H-equivariantly isomorphic σ-C^*-algebras $\hat{C}_0(\mathcal{P}(X), D)$. Even more, coarsely equivalent coarse spaces yield homotopy equivalent systems of spaces $\mathcal{P}(X)$ and hence homotopy equivalent σ-C^*-algebras $\hat{C}_0(\mathcal{P}(X), D)$ (see [4]). Hence the following definition is legitimate.

Definition 31. Let X be a coarse space with action of a finite group H. The *H-equivariant coarse* K*-theory of X with coefficents in D* is defined by

$$\mathrm{KX}^*_H(X, D) = \mathrm{K}_*\big(\hat{C}_0(\mathcal{P}(X), D) \rtimes H\big).$$

The following lemma is proved in [4].

Lemma 32. *The assignment $X \mapsto \mathrm{KX}^*_H(X, D)$ is functorial from the category of coarse spaces with isometric actions of H, and H-equivariant coarse maps, to the category of Abelian groups and Abelian group homomorphisms. That is, an equivariant coarse map $\phi\colon X \to Y$ induces a canonical homomorphism $\mathrm{KX}^*_H(Y, D) \to \mathrm{KX}^*_H(X, D)$ for any coefficient algebra D, and close maps induce the same map $\mathrm{KX}^*_H(Y, D) \to \mathrm{KX}^*_H(X, D)$.*

We can now define the coarse co-assembly map. Let X be a coarse space equipped with an isometric action of a finite group H, and let D be an H-C^*-algebra. Construct the inductive system $P_n(X)$ with injective, H-equivariant coarse equivalences $i_n\colon P_n(X) \to P_{n+1}(X)$ as above. Since the spaces $P_n(X)$ are themselves coarse spaces, we can form the algebras $\bar{\mathfrak{c}}^{\mathfrak{red}}(P_n(X), D) \rtimes H$ and $\mathfrak{c}^{\mathfrak{red}}(P_n(X), D) \rtimes H$. By functoriality of the various constructions involved, the maps i_n give rise to commutative diagrams with exact rows

$$\begin{tikzcd}
C_0(P_{n+1}(X), D) \rtimes H \arrow[r] \arrow[d] & \bar{\mathfrak{c}}^{\mathfrak{red}}(P_{n+1}(X), D) \rtimes H \arrow[r] \arrow[d] & \mathfrak{c}^{\mathfrak{red}}(P_{n+1}(X), D) \rtimes H \arrow[d, "\cong"] \\
C_0(P_n(X), D) \rtimes H \arrow[r] & \bar{\mathfrak{c}}^{\mathfrak{red}}(P_n(X), D) \rtimes H \arrow[r] & \mathfrak{c}^{\mathfrak{red}}(P_n(X), D) \rtimes H.
\end{tikzcd}$$

We have surjective maps on kernel and quotient. Hence the maps on $\bar{\mathfrak{c}}^{\mathfrak{red}}(\cdots)$ are also surjective by the Snake Lemma.

Lemma 33 ([14])**.** *Suppose that $\alpha_n\colon A_{n+1} \to A_n$ is an inverse system of C^*-algebras with surjective maps α_n for all n. Let J_n be ideals in A_n such that the restriction of α_n to J_{n+1} maps J_{n+1} surjectively onto J_n. Then the sequence*

$$0 \to \varprojlim J_n \to \varprojlim A_n \to \varprojlim A_n/J_n \to 0$$

is an exact sequence of σ-C^*-algebras.

We have already introduced the inverse limit $\hat{C}_0(\mathcal{P}(X), D) \rtimes H$ of the ideals $C_0(P_n(X), D)$ above. Since the maps on $\mathfrak{c}^{\mathfrak{red}}(P_n(X), D) \rtimes H$, are all isomorphisms, the inverse limit of this system is again isomorphic to $\mathfrak{c}^{\mathfrak{red}}(X, D)$. Let

$$\bar{\mathfrak{c}}^{\mathfrak{red}}(\mathcal{P}(X), D) \rtimes H \overset{\text{def}}{=} \varprojlim \bar{\mathfrak{c}}^{\mathfrak{red}}(P_n(X), D) \rtimes H.$$

Phillips shows in [14] how to associate to an exact sequence of σ-C^*-algebras a long exact sequence in K-theory. This is how we define the coarse co-assembly map:

Definition 34. Let X be a coarse space, let H be a finite group acting on X, and let D be an H-C^*-algebra. The *H-equivariant coarse co-assembly map with coefficients in D* is the connecting map

$$\mu^*_{D,H} \colon \mathrm{K}_{*+1}(\mathfrak{c}^{\mathfrak{red}}(X, D) \rtimes H) \to \mathrm{KX}^*_H(X, D)$$

associated to the exact sequence of σ-C^*-algebras

$$0 \to \hat{C}_0(\mathcal{P}(X), D) \rtimes H \to \bar{\mathfrak{c}}^{\mathfrak{red}}(\mathcal{P}(X), D) \rtimes H \to \mathfrak{c}^{\mathfrak{red}}(X, D) \rtimes H \to 0.$$

Now let G be a countable discrete group, equipped with its canonical coarse structure. Let (Σ_n) be an increasing sequence of finite subsets of G with $G = \bigcup \Sigma_n$ and $\Sigma_n = \Sigma_n^{-1}$ for all $n \in \mathbb{N}$ and $\Sigma_0 = \{1\}$. For each n, we get an entourage

$$E_n \overset{\text{def}}{=} \{(g_1, g_2) \in G \times G \mid g_1 g_2^{-1} \in \Sigma_n\}.$$

These entourages can be used to define the inductive system $\mathcal{P}(X)$ that occurs in the definition of the coarse K-theory of G. The entourage E_n is clearly invariant under right translation by G, so that the associated simplicial complex $P_n(G) = P_{E_n}(G)$ is a G-space. In addition, the action of G on $P_n(G)$ is proper and G-compact. Hence $|\mathcal{P}(G)|$ is a proper G-space. It is not locally compact because the corresponding simplicial complex is not locally finite. Nevertheless, it is a model for the universal proper G-space $\mathcal{E}G$ in the following sense:

Proposition 35. *For any proper G-space X, the space of G-equivariant maps $X \to |\mathcal{P}(G)|$ is naturally homeomorphic to the space of cut-off functions on X and hence contractible. Thus $\mathcal{P}(G)$ is a model for $\mathcal{E}G$.*

Proof. We view elements of $|\mathcal{P}(G)|$ as functions $f \colon G \to [0, 1]$ with compact support and $\sum f(g) = 1$. An equivariant map $h \colon X \to |\mathcal{P}(G)|$ is already determined by the map $h_* \colon X \to [0, 1]$, $h_*(x) = h(x)(1)$. This map is continuous, satisfies $\sum h_*(gx) = 1$ for all $x \in X$, and the support of h_* intersects each G-compact subset of X in a compact subset. That is, h_* is a cut-off function on X. Conversely, any cut-off function arises from a unique map $X \to |\mathcal{P}(G)|$. The space of cut-off functions on X is non-empty for any sufficiently regular, proper G-space X. It is convex and hence contractible. Hence there exists a unique map up to homotopy $X \to |\mathcal{P}(G)|$ for any sufficiently regular, proper G-space. That is, $|\mathcal{P}(G)|$ is universal. □

The above model for $\mathcal{E}G$ is a variant of the model constructed by G. Kasparov and G. Skandalis in [10]. Let now X be any second countable, proper G-space. We let (X_n) be an increasing sequence of G-compact subsets of X such that any G-compact subset of X is contained in X_n for some $n \in \mathbb{N}$, and we let $\hat{C}_0(X) \overset{\text{def}}{=} \hat{C}_0\big((X_n)\big)$. The same reasoning as for $\hat{C}_0(\mathcal{P}(X))$ identifies

$$\hat{C}_0(X) \cong \{f \colon X \to \mathbb{C} \mid f_{|Y} \in C_0(Y) \text{ for all } G\text{-compact subsets } Y \subseteq X\}.$$

Clearly, any continuous, G-equivariant map $X \to Y$ induces a $*$-homomorphism $\hat{C}_0(Y) \to \hat{C}_0(X)$. Thus equivariantly homotopic maps induce equivariantly homotopic $*$-homomorphisms. Since the universal proper G-space $\mathcal{E}G$ is uniquely determined up to G-equivariant homotopy equivalence, we obtain:

Lemma 36. *Let $H \subseteq G$ be a finite subgroup and let D be an H-C^*-algebra. Let $\mathcal{E}G$ be any second countable, universal proper G-space. Then the σ-C^*-algebras $\hat{C}_0(\mathcal{E}G, D) \rtimes H$ and $\hat{C}_0(\mathcal{P}(X), D) \rtimes H$ are homotopy equivalent.*

7. The Main Theorem

We are going to identify the coarse co-assembly map and the analytic co-assembly map with appropriate coefficients. We need two preparatory results. The first is well-known for C^*-algebras, and the proof for σ-C^*-algebras is exactly the same.

Lemma 37. *Let G be a discrete group and H a finite subgroup. Let A be a G-σ-C^*-algebra and B an H-σ-C^*-algebra.*

(1) The σ-C^*-algebras $A \otimes \operatorname{Ind}_H^G B$ and $\operatorname{Ind}_H^G(A \otimes B)$ are G-equivariantly isomorphic.
(2) The σ-C^*-algebras $\bigl(\operatorname{Ind}_H^G A\bigr) \rtimes G$ and $A \rtimes H$ are strongly Morita equivalent.

Lemma 38. *Let G be a discrete group, H a finite subgroup, and D an H-C^*-algebra. Then there is a canonical isomorphism*

$$\mathrm{K}_*\bigl(\hat{C}_0(\mathcal{E}G, D) \rtimes H\bigr) \cong \mathrm{K}_*\bigl(\hat{C}_0(\mathcal{E}G, \operatorname{Ind}_H^G D) \rtimes G\bigr).$$

Proof. Lemma 37 implies

$$\begin{aligned}\hat{C}_0(\mathcal{E}G, \operatorname{Ind}_H^G D) \rtimes G &\cong (\hat{C}_0(\mathcal{E}G) \otimes \operatorname{Ind}_H^G D) \rtimes G \\ &\cong \operatorname{Ind}_H^G(\hat{C}_0(\mathcal{E}G) \otimes D) \rtimes G \sim \hat{C}_0(\mathcal{E}G, D) \rtimes H,\end{aligned}$$

where $\cong$ denotes isomorphism and $\sim$ denotes strong Morita equivalence. The result follows. □

Lemma 39. *Let G be a discrete group and X a second countable, proper G-space, and let B be a G-C^*-algebra. Then there is a natural isomorphism*

$$\mathrm{RKK}^G_*(X; \mathbb{C}, B) \cong \mathrm{K}_*(\hat{C}_0(X, B) \rtimes G).$$

Proof. We check that both groups agree on the level of cycles after some standard simplifications for cycles for $\mathrm{RKK}^G(X; \mathbb{C}, B)$. Since $C_0(X, B)$ is a proper G-C^*-algebra, the reduced and full crossed products for $C_0(X, B)$ are equal. Moreover, the C^*-categories of G-equivariant Hilbert modules over $C_0(X, B)$ and of Hilbert modules over $C_0(X, B) \rtimes G$ are equivalent (see [12]). That is, any G-equivariant Hilbert module $\mathcal{E}$ over $C_0(X, B)$ corresponds to a Hilbert module $\tilde{\mathcal{E}}$ over $C_0(X, B) \rtimes G$. The correspondence is such that $\mathbb{B}(\tilde{\mathcal{E}})$ is naturally isomorphic to the C^*-algebra $\mathbb{B}(\mathcal{E})^G$ of G-equivariant, adjointable operators on $\mathcal{E}$. The compact operators on $\tilde{\mathcal{E}}$ correspond to the *generalized fixed point algebra* of $\mathbb{K}(\mathcal{E})$, which is the closed linear span of operators of the form $\sum_{g \in G} \alpha_g(|\xi\rangle\langle\eta|)$, where $\xi, \eta \in \mathcal{E}$ are *compactly supported* sections. (The support of ξ is the set of all $x \in X$ with $\xi_x \neq 0$.) More generally, if $T \in \mathbb{K}(\mathcal{E})$ has compact support in an appropriate sense, then $\sum_{g \in G} \alpha_g(T)$ belongs to the generalized fixed point algebra.

The cycles for $\mathrm{RKK}^G(X; \mathbb{C}, B)$ are pairs $(\mathcal{E}, F)$ where $\mathcal{E}$ is a (graded) G-equivariant Hilbert module over $C_0(X, B)$ and where $F \in \mathbb{B}(\mathcal{E})$ satisfies $F = F^*$, $-1 \le F \le 1$, F is odd in the graded case, $\phi(1 - F^2) \in \mathbb{K}(\mathcal{E})$ for all $\phi \in C_0(X)$, and F is G-equivariant. We can arrange F to be exactly equivariant (see [8, 11]) since X is a proper G-space. By our category equivalence, all this data may be rewritten in terms of a pair $(\tilde{\mathcal{E}}, \tilde{F})$, where $\tilde{\mathcal{E}}$ is a Hilbert module over $C_0(X, B) \rtimes G$ and $\tilde{F} \in \mathbb{B}(\tilde{\mathcal{E}})$ satisfies $\tilde{F} = \tilde{F}^*$, $-1 \le \tilde{F} \le 1$, and $\tilde{F}$ is odd in the graded case. There is an additional condition on $1 - \tilde{F}^2$ which we now identify.

For every G-invariant, closed subset $Y \subseteq X$, we define restrictions of $\mathcal{E}$ and $\tilde{\mathcal{E}}$ to Y by

$$\mathcal{E}_Y \overset{\text{def}}{=} \mathcal{E}/C_0(X \setminus Y) \cdot \mathcal{E}, \qquad \tilde{\mathcal{E}}_Y \overset{\text{def}}{=} \tilde{\mathcal{E}}/C_0\big((X \setminus Y)/G\big) \cdot \tilde{\mathcal{E}}.$$

We claim that $\phi(F^2-1) \in \mathbb{K}(\mathcal{E})$ for all $\phi \in C_0(X)$ if and only if the operator on $\tilde{\mathcal{E}}_Y$ induced by $1-\tilde{F}^2$ is compact for all G-compact subsets $Y \subseteq X$.

Assume first that $\phi(F^2-1) \in \mathbb{K}(\mathcal{E})$ for all $\phi \in C_0(X)$. Choose a G-compact subset $Y \subseteq X$. By the properness of the G-action, there exists a function $\phi \in C_c(X)$ with $\sum_{g\in G} \phi(xg) = 1$ for all $x \in Y$. Then $1-F^2$ and $\sum_{g\in G} \alpha_g(\phi(1-F^2))$ have the same restriction to Y. Since $\phi(1-F^2)$ is compact by hypothesis and has compact support, this sum belongs to $\mathbb{K}(\tilde{\mathcal{E}})$. Thus $(1-\tilde{F}^2)$ induces a compact operator on $\tilde{\mathcal{E}}_Y$ for all G-compact $Y \subseteq X$.

Suppose conversely that $(1-\tilde{F}^2)$ induces a compact operator on $\tilde{\mathcal{E}}_Y$ for all G-compact $Y \subseteq X$. Let $\phi \in C_c(X)$. Let Y be a G-compact subset containing the support of ϕ. Then $(1-\tilde{F}^2)|_Y$ belongs to the generalized fixed point algebra of $\mathcal{E}_Y$ and hence can be approximated by operators of the form $\sum_{g\in G}\alpha_g T$ for a finite rank operator T on $\mathcal{E}$ with compact support. Hence the function $g \mapsto \phi\alpha_g T$ has compact support, so that $\phi\sum_{g\in G}\alpha_g T$ is a compact operator on $\mathcal{E}$. Since these operators approximate $\phi(1-F^2)$, the latter operator is also compact.

Finally, we replace $C_0(X,B)$ by the σ-C^*-algebra $\hat{C}_0(X,B)$. A Hilbert module $\tilde{\mathcal{E}}$ over $C_0(X,B) \rtimes G$ automatically extends to a Hilbert module $\hat{\mathcal{E}} \overset{\text{def}}{=} \varprojlim \tilde{\mathcal{E}}_{X_n}$ over $\hat{C}_0(X,B) \rtimes G$. Conversely, any Hilbert module over $\hat{C}_0(X,B) \rtimes G$ is of this form. By definition, the algebra of compact operators on $\hat{\mathcal{E}}$ is $\varprojlim \mathbb{K}(\tilde{\mathcal{E}}_{X_n})$. Hence the above condition on $1-\tilde{F}^2$ is equivalent to $1-\tilde{F}^2 \in \mathbb{K}(\hat{\mathcal{E}})$. We conclude that

$$\mathrm{RKK}^G(X;\mathbb{C},B) \cong \mathrm{KK}(\mathbb{C},\hat{C}_0(X,B) \rtimes G) = \mathrm{K}(\hat{C}_0(X,B) \rtimes G),$$

because these groups can be defined by exactly the same cycles and homotopies. $\square$

Theorem 40. *Let G be a discrete group, H a finite subgroup of G, and D an H-C^*-algebra. Then there is a canonical isomorphism*

$$\mathrm{KX}^*_H(G,D) \cong \mathrm{RKK}^G(\mathcal{E}G;\mathbb{C},\mathrm{Ind}_H^G D)$$

such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{KK}^G_*(\mathbb{C},\mathrm{Ind}_H^G D) \arrow[r, "p^*_{\mathcal{E}G}"] \arrow[d, "\cong"] & \mathrm{RKK}^G_*(\mathcal{E}G;\mathbb{C},\mathrm{Ind}_H^G D) \arrow[d, "\cong"] \\
\mathrm{K}_{*+1}(\mathfrak{c}^{\mathfrak{red}}(G,D) \rtimes H) \arrow[r, "\mu^*_{D,H}"] & \mathrm{KX}^*_H(X,D).
\end{tikzcd} \tag{8}$$

Proof. The lemmas 39, 38, and 36 yield isomorphisms

$$\begin{aligned}\mathrm{RKK}^G(\mathcal{E}G;\mathbb{C},\mathrm{Ind}_H^G D) &\cong \mathrm{K}_*(\hat{C}_0(\mathcal{E}G,\mathrm{Ind}_H^G D) \rtimes G)\\ &\cong \mathrm{K}_*(\hat{C}_0(\mathcal{E}G,D) \rtimes H) \cong \mathrm{KX}^*_H(G,D).\end{aligned}$$

It remains to check that this makes our diagram commute. We identify cycles for $\mathrm{KK}^G_1(\mathbb{C},\mathrm{Ind}_H^G D)$ with certain elements $F \in \bar{\mathfrak{c}}^{\mathfrak{red}}(G,D) \rtimes H$ as in the proof of Theorem 30. Thus the isomorphism $\mathrm{KK}^G_1(\mathbb{C},\mathrm{Ind}_H^G D) \to \mathrm{K}_*(\mathfrak{c}^{\mathfrak{red}}(G,D) \rtimes H)$ maps $[F]$ to the class of the projection $P_F \overset{\text{def}}{=} ([F]-1)/2$ in $\mathrm{K}_0(\mathfrak{c}^{\mathfrak{red}}(G,D) \rtimes H)$.

Let $0 \to I \to E \to Q \to 0$ be an extension of σ-C^*-algebras and let $P \in Q$ be a projection. Let $\partial\colon \mathrm{K}_0(Q) \to \mathrm{K}_1(I)$ be the connecting map. To compute $\partial[P]$, we lift P to any $\bar{P} \in E$ and embed $E \subseteq \mathcal{M}(I)$. Then $\bar{P}^2 - \bar{P} \in I$, and $\partial[P] \in \mathrm{K}_1(I) \cong \mathrm{KK}_1(\mathbb{C},I)$ is represented by the cycle $(I, 2\bar{P}-1)$. We apply this to compute $\mu^*_{D,H}([P_F])$. It is easy to see that we can work with fixed point algebras instead of

crossed products for the computation. In particular, we have $\hat{C}_0(\mathcal{E}G, D_H \otimes \mathbb{K})^H \cong \hat{C}_0(\mathcal{E}G, D \otimes \mathbb{K}) \rtimes H$. Instead of lifting P_F, we may just as well lift $[F]$. Thus, we have to extend $F \in \bar{\mathfrak{c}}^{\mathfrak{red}}(G, D_H)^H$ to $\mathcal{E}G$. Choose an H-invariant continuous function $\phi\colon \mathcal{E}G \to \mathbb{R}_+$ such that (1) $S_Y \overset{\text{def}}{=} \operatorname{supp}\phi \cap Y$ is compact for all G-compact subsets $Y \subseteq \mathcal{E}G$, and (2) $\sum_{g\in G} \phi(xg) = 1$ for all $x \in \mathcal{E}G$. Let $L_Y \subseteq G$ be the set of all $g \in G$ with $S_Y\, g \cap S_Y \neq \emptyset$. We let

$$\bar{F}(x) \overset{\text{def}}{=} \sum_{g\in G} \phi(xg)\, F(g^{-1}).$$

If $x \in S_Y\, g$ for some $g \in G$, then $\bar{F}(x)$ is an average of $F(h)$ with $xh^{-1} \in S_Y$, so that $h \in L_Y^{-1} g$. Hence $\|\bar{F}|_{S_Y g} - F_g\|_\infty \to 0$ for $g \to \infty$. Thus $\bar{F}|_Y$ belongs to $\bar{\mathfrak{c}}^{\mathfrak{red}}(Y, D_H)^H$ for all G-compact Y, that is, $\bar{F} \in \bar{\mathfrak{c}}^{\mathfrak{red}}(\mathcal{E}G, D_H)^H$. The quotient map $\varinjlim \bar{\mathfrak{c}}^{\mathfrak{red}}(\mathcal{E}G, D_H)^H \to \mathfrak{c}^{\mathfrak{red}}(G, D_H)^H$ simply restricts a function on $\mathcal{E}G$ to any G-orbit in $\mathcal{E}G$. Hence $[\bar{F}] = [F]$ in $\mathfrak{c}^{\mathfrak{red}}(G, D_H)^H$. Thus $\mu^*_{D,H}([P_F])$ is represented by the Kasparov cycle $\bar{F}$ for $\mathrm{KK}(\mathbb{C}, \hat{C}_0(\mathcal{E}G, D_H \otimes \mathbb{K})^H) \cong \mathrm{KK}(\mathbb{C}, \hat{C}_0(\mathcal{E}G, D) \rtimes H)$.

Next we go around the diagram the other way. By definition, $p^*_{\mathcal{E}G}(\alpha)$ is represented by the Kasparov cycle $(C_0(\mathcal{E}G, \operatorname{Ind}_H^G(D \otimes \ell^2(H \times \mathbb{N})), F')$, where $F'f(x,g) \overset{\text{def}}{=} F_g f(x,g)$ and G acts on $\mathcal{E}G \times G$ by $h\cdot(x,g) = (xh^{-1}, hg)$. The same formula makes sense if we replace C_0 by $\hat{C}_0$. To map this to $\mathrm{K}_0(\hat{C}_0(\mathcal{E}G, D) \rtimes H)$, we first have to make F' G-equivariant. We use the same cut-off function ϕ as above to average F':

$$F''f(x,g) \overset{\text{def}}{=} \sum_{h\in G} F_{h^{-1}g}\phi(xh)\ \cdot f(x,g)$$

for all $x \in \mathcal{E}G$, $g \in G$. This is a compact perturbation of F' that is G-equivariant. Thus F'' defines a multiplier of the generalized fixed point algebra of the Hilbert module $\hat{C}_0(\mathcal{E}G, \operatorname{Ind}_H^G(D \otimes \ell^2(H \times \mathbb{N})))$. Restriction to $\mathcal{E}G \times \{1\} \subseteq \mathcal{E}G \times G$ identifies this generalized fixed point algebra with $\hat{C}_0(\mathcal{E}G, D_H \otimes \mathbb{K})^H \cong \hat{C}_0(\mathcal{E}G, D \otimes \mathbb{K}) \rtimes H$. The composition of isomorphisms

$$\begin{aligned}\mathrm{RKK}^G(\mathcal{E}G; \mathbb{C}, \operatorname{Ind}_H^G D) &\cong \mathrm{K}_0(\hat{C}_0(\mathcal{E}G, \operatorname{Ind}_H^G D) \rtimes G)\\ &\cong \mathrm{K}_0(\mathbb{C}, \hat{C}_0(\mathcal{E}G, D \otimes \mathbb{K}) \rtimes H) \overset{\text{def}}{=} \mathrm{KK}_0(\mathbb{C}, \hat{C}_0(\mathcal{E}G, D \otimes \mathbb{K}) \rtimes H)\end{aligned}$$

constructed above sends F' to the Kasparov cycle $(\hat{C}_0(\mathcal{E}G, D \otimes \mathbb{K}) \rtimes H, F''|_{\mathcal{E}G\times\{1\}})$. The reason is that Green's Imprimitivity Theorem is proved using the same manipulations of generalized fixed point algebras that we used above to view F'' as a multiplier of $\hat{C}_0(\mathcal{E}G, D \otimes \mathbb{K}) \rtimes H$. By construction, $F''|_{\mathcal{E}G\times\{1\}} = \bar{F}$. This means that the diagram commutes on $\mathrm{KK}_1^G(\mathbb{C}, \operatorname{Ind}_H^G D)$. The other parity can be treated by a similar argument, or by replacing D by $C_0(\mathbb{R}, D)$. □

Combining the above with Theorem 20 and Proposition 16, we obtain:

Theorem 41. *Let G be a discrete, strongly geometrically finite group. The following are equivalent:*

(1) the H-equivariant coarse co-assembly map with coefficients in D is an isomorphism for every finite subgroup H of G and every H-C^*-algebra D;
(2) the H-equivariant coarse co-assembly map with coefficients in $C_0(\mathbb{N})$ is an isomorphism for every finite subgroup H of G.
(3) G has a γ-element.

Moreover, (3) implies (1) and (2) for an arbitrary discrete group.

Theorem 42. *Let G be a discrete group with a G-finite model for $\mathcal{E}G$. Then G has a γ-element if and only if the H-equivariant coarse co-assembly map with trivial coefficients is an isomorphism for every finite subgroup H of G.*

If G is torsion free, then the only finite subgroup is the trivial group. Since the coarse co-assembly map is an invariant of the coarse structure, we obtain:

Corollary 43. *If G is a torsion free, geometrically finite group, then the existence or non-existence of a γ-element for G is a coarse invariant of G.*

References

[1] Paul Baum, Alain Connes, and Nigel Higson, *Classifying space for proper actions and K-theory of group C^*-algebras*, C^*-Algebras: 1943–1993 (San Antonio, TX, 1993), Contemp. Math., vol. 167, Amer. Math. Soc., Providence, RI, 1994, pp. 240–291. MR 96c:46070

[2] Bruce Blackadar, *K-theory for operator algebras*, 2nd ed., Mathematical Sciences Research Institute Publications, vol. 5, Cambridge University Press, Cambridge, 1998, ISBN 0-521-63532-2. MR 99g:46104

[3] Alain Connes, *Noncommutative geometry*, Academic Press Inc., San Diego, CA, 1994, ISBN 0-12-185860-X. MR 95j:46063

[4] Heath Emerson and Ralf Meyer, *Dualizing the Baum-Connes coarse assembly map*, 2003 (in preparation).

[5] Erik Guentner, Nigel Higson, and Jody Trout, *Equivariant E-theory for C^*-algebras*, Mem. Amer. Math. Soc. **148** (2000), viii+86. MR 2001c:46124

[6] N. Higson, *Bivariant K-theory and the Novikov conjecture*, Geom. Funct. Anal. **10** (2000), 563–581. MR 2001k:19009

[7] Nigel Higson and John Roe, *Analytic K-homology*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000, ISBN 0-19-851176-0, Oxford Science Publications. MR 2002c:58036

[8] G. G. Kasparov, *Equivariant KK-theory and the Novikov conjecture*, Invent. Math. **91** (1988), 147–201. MR 88j:58123

[9] G. G. Kasparov and G. Skandalis, *Groups acting on buildings, operator K-theory, and Novikov's conjecture*, K-Theory **4** (1991), 303–337. MR 92h:19009

[10] Gennadi Kasparov and Georges Skandalis, *Groups acting properly on "bolic" spaces and the Novikov conjecture*, Ann. of Math. (2) **158** (2003), 165–206. 1 998 480

[11] Ralf Meyer, *Equivariant Kasparov theory and generalized homomorphisms*, K-Theory **21** (2000), 201–228. MR 2001m:19013

[12] ———, *Generalized fixed point algebras and square-integrable groups actions*, J. Funct. Anal. **186** (2001), 167–195. MR 2002j:46086

[13] Ralf Meyer and Ryszard Nest, *The Baum-Connes conjecture via localization of categories*, 2003 (in preparation).

[14] N. Christopher Phillips, *Representable K-theory for σ-C^*-algebras*, K-Theory **3** (1989), 441–478. MR 91k:46082

[15] John Roe, *Coarse cohomology and index theory on complete Riemannian manifolds*, Mem. Amer. Math. Soc. **104** (1993), x+90. MR 94a:58193

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